

## On Csákány's problem concerning affine spaces

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*Dedicated to Professor Béla Csákány on his 60th birthday*

In [1] B. CSÁKÁNY proved that for any prime  $p$  an algebra  $\mathfrak{A}=(A, f)$  where  $f$  is at most 4-ary, is equivalent to an affine space over  $\text{GF}(p)$  if and only if

$$(*) \quad p_n(\mathfrak{A}) = \frac{1}{p} ((p-1)^n - (-1)^n)$$

for all  $n \geq 4$ , and

$$(**) \quad \text{There exists no subalgebra } B \text{ of } A \text{ with } 1 < \text{card } B < p$$

(in this case, the formula  $(*)$  is valid for all  $n \geq 0$ ). In this connection, he posed the problem whether the condition  $(**)$  can be dropped for some or all  $p$ . Earlier G. GRÄTZER and R. PADMANABHAN [11] showed that if  $\mathfrak{A}$  is a groupoid and  $p=3$ , then actually the single condition  $(*)$  is sufficient. Our result is a further step in this problem

**Theorem.** *If  $G$  is a groupoid, then  $G$  is equivalent to an affine space over  $\text{GF}(5)$  if and only if  $p_n(G) = \frac{1}{5}(4^n - (-1)^n)$  for all  $n \geq 0$ .*

(Of course, as in CSÁKÁNY's result [1], by an affine space we mean a nontrivial, i.e. containing more than one element, affine space.) In the sequel equivalent algebras are treated as identical and "an algebra" means always "a nontrivial algebra". Our terminology and notation are standard (see in [9] and [10]).

To prove our theorem we need among others the following results:

**Fact 1** (Theorem 4.1 of [5]). If  $(G, \cdot)$  is a nonmedial commutative idempotent groupoid, then

$$p_n(G, \cdot) \geq \frac{7}{8} n! \quad \text{for all } n \geq 5.$$

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Recall that the groupoid  $(G, \cdot)$  is medial if  $(G, \cdot)$  satisfies  $(xy)(uv) = (xu)(yv)$  for all  $x, y, u, v \in G$ .

Fact 2 (cf. [6]). Let  $(G, \cdot)$  be a medial idempotent groupoid with  $\text{card } G > 1$ . Then  $p_2(G, \cdot) = 3$  if and only if  $(G, \cdot)$  is either a (nontrivial) affine space over  $\text{GF}(5)$  or a nontrivial Plonka sum of some affine spaces over  $\text{GF}(3)$  which are not all singletons (for the definition of a Plonka sum see [12]).

Fact 3. If  $(A, +, \cdot)$  is a proper commutative idempotent algebra of type  $(2, 2)$  satisfying (cf. [7], also [8])

$$(x+z)z = (x+z)y \quad (\text{or the dual})$$

then  $(A, +, \cdot)$  is polynomially infinite, i.e.,  $p_n(A, +, \cdot)$  is infinite for all  $n \geq 2$ .

A proper algebra here means that  $x+y$  and  $xy$  act on  $A$  differently.

Fact 4 (cf. Theorem II of [5]). Let  $(G, \cdot)$  be a commutative idempotent groupoid. Then  $(G, \cdot)$  is a nontrivial Plonka sum of affine spaces over  $\text{GF}(3)$  being not all one-element if and only if  $p_n(G, \cdot) = 3^{n-1}$  for all  $n$ .

**1. General remarks.** First observe that if an algebra  $\mathfrak{A}$  satisfies  $(*)$  for  $p=5$ , then  $\mathfrak{A}$  represents the sequence  $\langle 0, 1, 3, 13 \rangle$ , i.e.,  $\mathfrak{A}$  is an idempotent algebra satisfying

$$p_2(\mathfrak{A}) = 3 \quad \text{and} \quad p_3(\mathfrak{A}) = 13.$$

Lemma 1.1. *If  $(A, F)$  represents the sequence*

$$\langle 0, 1, 3, 13 \rangle$$

*then  $(A, F)$  contains as a reduct a proper idempotent algebra  $(A, +, *)$  of type  $(2, 2)$  such that  $+$  is commutative and  $*$  is noncommutative. Moreover the polynomials  $x+y$ ,  $x*y$  and  $y*x$  are the only essentially binary polynomials over  $(A, F)$ .*

Proof. Since  $p_2(A, F)$  is odd we infer that the algebra  $(A, F)$  contains at least one commutative and essentially binary operation, say,  $+$ . If all binary polynomials over  $(A, F)$  are commutative, then we infer that  $(A, F)$  contains as a reduct a proper commutative idempotent algebra  $(A, +, \cdot, \circ)$  of type  $(2, 2, 2)$ . Examining the symmetry groups of the following essentially ternary polynomials:  $(x+y)+z$ ,  $(xy)z$ ,  $(x \circ y) \circ z$ ,  $(x+y)z$ ,  $xy+z$ ,  $(x+y) \circ z$ ,  $x \circ y + z$ ,  $xy \circ z$  and  $(x \circ y)z$  and using the Fact 3 we deduce that  $p_3(A, +, \cdot, \circ) \geq 21$  which is impossible. (Recall that an algebra  $(A, \{f_i\}_{i \in T})$  of type  $\tau = (n_i)_{i \in T}$  is called proper if the mapping  $i \rightarrow n_i$  is one-to-one and every operation  $f_i$  is essentially  $n_i$ -ary provided  $n_i \geq 1$ , cf. [5].)

**Lemma 1.2.** *If an algebra  $\mathfrak{A}=(A, F)$  satisfies  $(*)$  for some  $p \geq 3$  and all  $n \geq 0$ , then  $\mathfrak{A}$  contains at least one commutative idempotent binary polynomial, say,  $+$  and each every such a polynomial is medial.*

**Proof.** The first statement is clear since  $p_2(\mathfrak{A})=p-2$  and hence  $p_2(\mathfrak{A})$  is an odd number. Assume now that  $(A, +)$  is nonmedial. Thus  $(A, +)$  is a non-medial commutative idempotent groupoid (being a reduct of  $\mathfrak{A}$ ). Applying Fact 1 we get

$$\frac{(p-1)^n - (-1)^n}{p} = p_n(\mathfrak{A}) \geq p_n(A, +) \geq \frac{7}{8}n!$$

for all  $n \geq 5$ . This yields

$$\frac{n!}{(p-1)^n - (-1)^n} \leq \frac{8}{7p}$$

for all  $n \geq 5$  which is impossible. This completes the proof of the lemma.

**Proposition 1.3.** *Let  $(G, \cdot)$  be a commutative groupoid. Then  $(G, \cdot)$  is a nontrivial affine space over  $\text{GF}(5)$  if and only if  $(G, \cdot)$  satisfies  $(*)$  for  $p=5$  and all  $n \geq 0$ .*

**Proof.** It is clear that  $(G, \cdot)$  is a nontrivial affine space over  $\text{GF}(5)$ , then  $p_n(G, \cdot) = \frac{4^n - (-1)^n}{5}$  for all  $n$  (see e.g., [1]).

If  $(G, \cdot)$  is a commutative groupoid such that  $p_n(G, \cdot) = \frac{4^n - (-1)^n}{5}$  for all  $n$ , then using Lemma 1.2 we infer that  $(G, \cdot)$  is a medial commutative idempotent groupoid. Since  $p_2(G, \cdot)=3$  and  $(G, \cdot)$  satisfies  $(*)$  for all  $n$  we infer, applying Fact 2 and Fact 4 that  $(G, \cdot)$  is an affine space over  $\text{GF}(5)$ .

**Lemma 1.4.** *If an idempotent algebra  $\mathfrak{A}=(A, F)$  with  $p_2(\mathfrak{A})>1$  contains as a reduct a Steiner quasigroup  $(A, +)$ , then  $p_2(\mathfrak{A}) \geq 5$ .*

**Proof.** Since  $p_2(\mathfrak{A})>1$  we infer that  $\mathfrak{A}$  contains as a reduct a proper binary idempotent algebra  $(A, +, \cdot)$  of type  $(2, 2)$  such that  $(A, +)$  is a Steiner quasigroup. If  $xy$  is commutative then the following polynomials

$$x+y, \quad xy, \quad x \circ y = (x+y) + (xy), \quad x * y = xy + y \quad \text{and} \quad y * x$$

are essentially binary and pairwise distinct.

Assume now that  $\cdot$  is noncommutative. Take into account the polynomial  $x * y = xy + y$ . It is easy to prove that  $x * y$  is essentially binary and different from the polynomials  $x+y$ ,  $xy$  and  $yx$ . If  $x * y \neq y * x$ , then we clearly get  $p_2(\mathfrak{A}) \geq 5$ .

Suppose that  $x * y = y * x$ . First observe that  $x * y \neq x + y$ . Further the polynomials

$$x + y, \quad x * y, \quad x \square y = (x + y) + x * y, \quad xy \quad \text{and} \quad yx$$

yield five essentially binary and pairwise distinct polynomials and hence  $p_2(\mathfrak{A}) \cong 5$ .

**Lemma 1.5.** *If  $(G, \cdot)$  is a noncommutative groupoid satisfying  $(*)$  for  $p=5$  and all  $n \geq 0$ , and  $(G, \cdot)$  is not (polynomially) equivalent to a commutative groupoid, then the unique commutative polynomial  $+$  over  $(G, \cdot)$  is a semilattice polynomial.*

**Proof.** Let us add that the uniqueness of the polynomial  $+$  follows from Lemma 1.1. Consider the reduct  $(G, +)$ . Put  $x \circ y = x + 2y$  (in general,  $xy^k$  stands for  $(\dots(xy) \cdot \dots)y$  where  $y$  occurs  $k$ -times and  $x + ky$  in the commutative case respectively). According to Theorem 1 of [1] we see that  $x \circ y \neq y$ . If  $x \circ y = x$ , then  $(G, +)$  is a Steiner quasigroup and then applying Lemma 1.4 we get  $p_2(G, \cdot) \cong 5$ , a contradiction. If  $x \circ y$  is commutative, then  $x \circ y = x + y$  and hence applying Lemma 1.2 we deduce that  $(G, +)$  is medial. According to Theorem 8 of [4] the groupoid  $(G, +)$  is a semilattice. If  $x \circ y$  is noncommutative (of course, essentially binary), then either  $x \circ y = xy$  or  $x \circ y = yx$ . Both cases prove that the groupoids  $(G, \cdot)$  and  $(G, +)$  are polynomially equivalent which contradicts the assumption. This completes the proof of the lemma.

**2. Noncommutative idempotent groupoids.** In this section we prove the theorem for the noncommutative case. We start with

**Lemma 2.1.** *If  $(G, \cdot)$  is a noncommutative idempotent groupoid having a commutative binary polynomial, then the following polynomials*

$$f(x, y, z) = (xy)z \quad \text{and} \quad g(x, y, z) = x(yz)$$

*are different and essentially ternary.*

**Proof.** Since  $(G, \cdot)$  contains a commutative binary polynomial we infer that  $(G, \cdot)$  is not a diagonal semigroup. Applying Lemma 3 of [2] we deduce that at least one of the polynomials  $f$  and  $g$  is essentially ternary. Further without loss of generality we may assume that  $f$  is not essentially ternary and  $g$  is essentially ternary. Since  $(G, \cdot)$  contains a commutative polynomial we infer that  $xy$  is essentially binary, i.e.,  $(G, \cdot)$  is proper. Thus we infer that  $(G, \cdot)$  satisfies either

$$(xy)z = xz \quad \text{or} \quad (xy)z = yz$$

If  $(G, \cdot)$  satisfies  $(xy)z = xz$ , then  $(G, \cdot)$  also satisfies the identities  $xy = (xy)y = x(xy)$  and  $x = (xy)x$  and every binary polynomial  $p(x, y)$  over  $(G, \cdot)$  is of the form:

$$x, y, xy, yx, y(xy), x(yx), x(y(xy)), y(x(yx)) \quad \text{and so on.}$$

If  $p(x, y) = p(y, x)$  holds in  $(G, \cdot)$ , then using the identity  $(xy)z = xz$  we get  $xz = yz$  which proves that  $(G, \cdot)$  is improper — a contradiction.

If the groupoid  $(G, \cdot)$  satisfies  $(xy)z = yz$ , then the proof runs similarly and will be omitted. To complete the proof one can easily show that there are no non-commutative idempotent semigroups with a commutative binary polynomial:

**Lemma 2.2.** *If  $(G, \cdot)$  is a noncommutative idempotent groupoid having a semi-lattice polynomial, say,  $+$  and the symmetry groups of the polynomials  $f$  and  $g$  are trivial, then  $p_3(G, \cdot) \cong 19$ .*

**Proof.** According to the preceding lemma we infer that  $f$  and  $g$  are essentially ternary and different. Consider now the following polynomials

$$(xy)z, \quad x(yz), \quad (x+y)z, \quad z(x+y) \quad \text{and} \quad x+y+z.$$

It is routine to prove that all these polynomials are essentially ternary and consequently permuting variables in them we get 19 different essentially ternary polynomials, as required.

**Lemma 2.3.** *If  $(G, \cdot)$  is a noncommutative idempotent groupoid satisfying  $(*)$  for  $p=5$  and all  $n$  such that  $(G, \cdot)$  is not polynomially equivalent to a commutative groupoid, then either the symmetry group of  $f$  is nontrivial or the symmetry group of  $g$  is nontrivial.*

**Proof.** An immediate consequence of Lemmas 1.5 and 2.2.

**Lemma 2.4.** *If  $(G, \cdot)$  is a proper noncommutative idempotent groupoid such that the symmetry group of the polynomial*

$$f(x, y, z) = (xy)z$$

*is nontrivial, then  $(G, \cdot)$  satisfies either*

$$(xy)z = (zy)x \quad \text{or} \quad (xy)z = (yx)z \quad \text{or} \quad (xy)z = (xz)y.$$

*(The same is true for  $g(x, y, z) = x(yz)$ .)*

**Proof.** Trivial since the identity  $(xy)z = (yz)x$  proves that  $(G, \cdot)$  is a semi-lattice.

**Proposition 2.5.** *Let  $(G, \cdot)$  be a noncommutative idempotent groupoid satisfying*

$$(xy)z = (zy)x \quad (\text{or the dual}).$$

*Then  $p_2(G, \cdot) = 3$  if and only if  $(G, \cdot)$  is a nontrivial affine space over  $\text{GF}(5)$ .*

**Proof.** It is clear that a nontrivial affine space over  $\text{GF}(5)$ , i.e., a groupoid  $(G, \cdot)$  where  $xy=2x+4y$  and  $(G, +)$  is an abelian group of exponent 5, satisfies  $p_2(G, \cdot)=3$  and  $(G, \cdot)$  satisfies  $(xy)z=(zy)x$ .

Assume now that  $p_2(G, \cdot)=3$ . It is easy to see that the identity  $(xy)z=(zy)x$  implies the medial law for the groupoid  $(G, \cdot)$ . Using Fact 2 we infer that  $(G, \cdot)$  is either a nontrivial affine space over  $\text{GF}(5)$  or a nontrivial Plonka sum of some affine spaces over  $\text{GF}(3)$  being not all one-element. The second algebra is a commutative idempotent groupoid which cannot be polynomially equivalent to a noncommutative groupoid. This follows from the fact that the only noncommutative binary polynomial in this groupoid is a  $P$ -function, but for  $P$ -functions we have  $p_2 \leq 2$  (for details see [12]). Thus we have proved that  $(G, \cdot)$  is an affine space over  $\text{GF}(5)$  which completes the proof.

**Proposition 2.6.** *If  $(G, \cdot)$  is a noncommutative idempotent groupoid satisfying*

$$(xy)z = (yx)z \quad (\text{or the dual})$$

*then  $p_2(G, \cdot) \neq 3$ .*

**Proof.** The assertion is obvious for improper groupoids.

First, we prove that if  $(G, \cdot)$  is a proper such groupoid, then the polynomial  $x \circ y = (xy)y$  is essentially binary and noncommutative.

If  $(xy)y=x$ , then  $(G, \cdot)$  is right cancellative and the identity  $(xy)z=(yx)z$  gives the commutativity of  $\cdot$ , a contradiction.

If  $(xy)y=y$  holds, then we obtain

$$xy = (x(xy))(xy) = ((yx)x)(xy) = x(xy).$$

Hence we get

$$y = (xy)y = (x(xy))y = ((yx)x)y = xy.$$

Thus  $xy=y$  which is impossible.

Assume now that  $(xy)y=(yx)x$  and denote  $(xy)y$  by  $x+y$ . Compute the polynomial  $xy+yx$ . We have

$$xy+yx = ((xy)(yx))(yx) = ((yx)(yx))(yx) = yx.$$

Thus  $(G, \cdot)$  is a commutative groupoid, a contradiction.

If  $x \circ y$  is essentially binary, noncommutative and  $x \circ y \notin \{xy, yx\}$ , then  $p_2(G, \cdot) \geq 4$  and therefore  $p_2(G, \cdot) \neq 3$ . Further assume that  $x \circ y = xy$ . Then we have  $xy=(xy)y=(yx)y$ . Putting  $yx$  for  $y$  in  $xy=(xy)y$  we get

$$x(yx) = (x(yx))(yx) = ((yx)x)(yx) = yx.$$

Analogously we get  $xy = x(xy)$ . This proves that

$$xy = (xy)y = (yx)y = x(xy) = y(xy)$$

and consequently  $p_2(G, \cdot) = 2$ , as required.

Similarly one proves that if  $(G, \cdot)$  satisfies  $(xy)y = yx$ , then  $(G, \cdot)$  also satisfies

$$xy = (yx)x = x(xy) = (xy)x = x(yx)$$

and therefore  $p_2(G, \cdot) = 2$ . The proof is completed.

Now we deal with the last identity appearing in Lemma 2.4, namely the identity  $(xy)z = (xz)y$  (the dual identity i.e.,  $x(yz) = y(xz)$  will be omitted in our considerations).

**Lemma 2.7.** *If  $(G, \cdot)$  is a proper noncommutative groupoid satisfying*

$$(xy)z = (xz)y,$$

*then the polynomial  $x \circ y = x(xy)$  is noncommutative and different from  $y$  and  $yx$ .*

**Proof.** If  $x(xy) = y(yx)$  holds in  $(G, \cdot)$ , then

$$x(xy) = (xx)(xy) = (x(xy))x = (y(yx))x = (yx)(yx) = yx.$$

Thus we get  $x(xy) = yx$  which proves that  $(G, \cdot)$  is commutative, a contradiction.

If  $x(xy) = y$ , then  $y = yy = (x(xy))y = xyxy = xy$ , again a contradiction. If  $x(xy) = yx$ , then

$$xy = (xy)(xy) = (x(xy))y = (yx)y = yx.$$

Thus  $xy = yx$  which is impossible.

**Lemma 2.8.** *There is no (noncommutative) idempotent groupoid  $(G, \cdot)$  satisfying  $p_2(G, \cdot) = 3$  and the identities*

$$(xy)z = (xz)y \quad \text{and} \quad x(xy) = x.$$

**Proof.** First we prove that the groupoid  $(G, \cdot)$  satisfies either

$$(xy)y = x \quad \text{or} \quad (xy)y = xy.$$

Indeed, if  $(xy)y = y$ , then

$$yx = ((xy)y)x = ((xy)x)y = ((xx)y)y = y$$

which is impossible. If  $(xy)y = yx$ , then

$$xy = (yx)x = ((xy)y)x = (xy)y = yx$$

which gives  $xy = yx$ , a contradiction.

If  $(xy)y=(yx)x$ , then putting  $yx$  for  $x$  we get

$$y = y(yx) = (y(yx))(yx) = ((yx)y)y = (yx)y = yx,$$

which is impossible. Hence we proved that  $(G, \cdot)$  satisfies either

$$(xy)y = x \quad \text{or} \quad (xy)y = xy.$$

Assume that  $(G, \cdot)$  satisfies  $(xy)y=x$ . Consider the polynomial  $x*y=x(yx)$ . If  $x*y=y*x$ , then

$$x = (x(yx))(yx) = (y(xy))(yx) = (y(yx))(xy) = y(xy).$$

Hence  $y(xy)=x$ , a contradiction.

Further it is easy to see that

$$x(yx) \neq y \quad \text{and} \quad x(yx) \neq yx.$$

According to the assumption  $p_2(G, \cdot)=3$  we infer that  $(G, \cdot)$  satisfies either

$$x(yx) = x \quad \text{or} \quad x(yx) = xy.$$

If so, then in both cases we get  $p_2(G, \cdot)=2$  which contradicts the assumption.

To complete the proof we must consider one more case, namely, the groupoid  $(G, \cdot)$  satisfies

$$(xy)z = (xz)y, \quad x(xy) = x \quad \text{and} \quad (xy)y = xy.$$

As above considering the polynomial  $x*y=x(yx)$  one proves that  $x*y$  is noncommutative and therefore the polynomial  $x*y$  is one of the following polynomials:  $x, y, xy, yx$ . In any case one can easily check that the considered groupoid satisfies  $p_2(G, \cdot)=2$  which is impossible. The proof of the lemma is completed.

**Lemma 2.9.** *Let  $(G, \cdot)$  be a proper noncommutative idempotent groupoid satisfying  $(xy)z=(xz)y$ . Then  $p_2(G, \cdot)=3$  if and only if  $(G, \cdot)$  satisfies the identities*

$$xy = x(xy) \quad \text{and} \quad x(yx) = y(xy).$$

*Moreover if an idempotent groupoid satisfies*

$$(xy)z = (xz)y, \quad xy = x(xy) \quad \text{and} \quad x(yx) = y(xy),$$

*then the polynomial  $x+y=x(yx)$  is a near-semilattice polynomial (i.e.,  $x+x=x$ ,  $x+y=y+x$  and  $x+y=(x+y)+y$ ; cf. [5]).*

**Proof.** Let  $p_2(G, \cdot)=3$ . Consider the polynomial  $x \circ y = x(xy)$ . Applying Lemma 2.7 we infer that  $(G, \cdot)$  satisfies either

$$x(xy) = x \quad \text{or} \quad x(xy) = xy.$$

According to Lemma 2.8, the first case cannot occur. Thus  $(G, \cdot)$  satisfies  $x(xy)=$



$=xy$ . Consider the polynomial  $x+y=x(yx)$ . If  $x+y \in \{x, y, xy, yx\}$ , then one gets  $p_2(G, \cdot)=2$ , a contradiction. If  $x+y$  is essentially binary noncommutative and different from  $xy, yx$ , then clearly  $p_2(G, \cdot) \geq 4$  which contradicts the assumption. Thus we have proved that  $x+y=y+x$ . Further we have

$$\begin{aligned}(x+y)+y &= x(yx)+y = (x(yx))(y(x(yx))) = \\ &= (x(yx))(y(y(xy))) = (x(yx))(y(xy)) = x+y.\end{aligned}$$

Hence  $x+y=(x+y)+y$  which proves that  $(G, +)$  is a near-semilattice.

Assume now that  $(G, \cdot)$  is noncommutative idempotent, satisfying  $xy=x(xy)$ ,  $x(yx)=y(xy)$ , and  $(xy)z=(xz)y$ . Since  $xy=(xy)y=(xy)x=x(xy)$  and  $(xy)(yx)=x(yx)$  we infer that (in a proper noncommutative groupoid) we have  $p_2(G, \cdot)=3$ . This completes the proof of the lemma.

**Lemma 2.10.** *If  $(G, \cdot)$  is an idempotent groupoid satisfying  $(xy)z=(xz)y$  and  $p_2(G, \cdot)=3$ , then the symmetry group of the polynomial  $g(x, y, z)=x(yz)$  is trivial.*

**Proof.** It is clear that  $g$  does not admit any cycle of its variables ( $(G, \cdot)$  is not a semilattice). If  $(G, \cdot)$  satisfies  $x(yz)=x(zy)$ , then using Proposition 2.6 we infer that  $p_2(G, \cdot) \neq 3$ .

If  $x(yz)=z(yx)$  holds in  $(G, \cdot)$ , then we obtain

$$xy = (xy)(xy) = (x(xy))y = (y(xx))y = (yx)y = yx.$$

Thus  $xy=yx$  which proves that  $(G, \cdot)$  is a semilattice, a contradiction. Assume now that  $(G, \cdot)$  satisfies  $x(yz)=y(xz)$ . Applying Lemma 2.9 we get  $x(yx)=y(xy)$  and hence using the identity  $x(yz)=y(xz)$  we get  $xy=yx$ , a contradiction.

**Proposition 2.11.** *If an idempotent groupoid  $(G, \cdot)$  satisfies  $(xy)z=(xz)y$  (or the dual identity) and  $p_2(G, \cdot)=3$ , then  $p_3(G, \cdot) \geq 16$ .*

**Proof.** According to Lemma 2.1 the polynomials  $f(x, y, z)=(xy)z$  and  $g(x, y, z)=x(yz)$  are essentially ternary and different. Applying Lemma 2.9 we see that  $x+y=x(yx)$  is a near-semilattice polynomial. It is clear that  $(G, \cdot)$  is a proper noncommutative idempotent groupoid and further the polynomials

$$q_1 = (x+y)z \quad \text{and} \quad q_2 = z(x+y)$$

are essentially ternary and their symmetry groups are of order 2. Consider now the following essentially ternary polynomials over  $(G, \cdot)$ :

$$f = (xy)z, \quad g = x(yz), \quad q_1 = (x+y)z, \quad q_2 = z(x+y) \quad \text{and} \quad s = (x+y)+z.$$

By the assumption and Lemma 2.10 we see that  $\text{card } G(f)=2$  and  $\text{card } G(g)=1$ .

We also have  $\text{card } G(q_1) = \text{card } G(q_2) = 2$ . Further observe that

$$(xy)z \neq x(y+z) \quad \text{and} \quad (xy)z \neq (y+z)x.$$

Indeed, if  $(xy)z = x(y+z)$ , then

$$xy = (xy)x = x(x+y) = x(x(yx)) = x(yx) = x+y$$

which proves that  $(G, \cdot)$  is commutative, a contradiction (we use also the identity  $x(xy) = xy$ , see Lemma 2.9). The proof of the inequality  $(xy)z \neq (y+z)x$  runs similarly. Further for the groupoid  $(G, \cdot)$  we have

$$\begin{aligned} p_3(G, \cdot) &\cong \frac{3!}{\text{card } G(f)} + \frac{3!}{\text{card } G(g)} + \frac{3!}{\text{card } G(q_1)} + \frac{3!}{\text{card } G(q_2)} + \\ &\quad + \frac{3!}{\text{card } G(s)} \cong 3 + 6 + 3 + 3 + 1 = 16 \end{aligned}$$

which finishes the proof of the lemma.

**3. The proof of the Theorem.** In this section we prove the theorem. First if  $(G, \cdot)$  is a nontrivial affine space over  $\text{GF}(5)$ , then clearly using the formula from [1] we see

$$p_n(G, \cdot) = \frac{4^n - (-1)^n}{5}$$

for all  $n$  (see also in [9]).

Let now  $(G, \cdot)$  satisfy  $(*)$  for all  $n$  and  $p=5$ .

If  $(G, \cdot)$  is commutative, then the proof follows from Proposition 1.3.

If  $(G, \cdot)$  is noncommutative but the groupoid  $(G, \cdot)$  is polynomially equivalent to a commutative groupoid, then the proof again follows from Proposition 1.3.

Assume that  $(G, \cdot)$  is a (proper) noncommutative idempotent groupoid being not polynomially equivalent to a commutative groupoid. Then applying Lemma 1.5 we infer that  $(G, \cdot)$  contains a semilattice polynomial, say,  $+$ .

Consider now the following polynomials

$$\begin{aligned} s &= (x+y)+z, \quad f = (xy)z, \quad g = x(yz) \\ q_1 &= (x+y)z \quad \text{and} \quad q_2 = z(x+y). \end{aligned}$$

All these polynomials are essentially ternary (see Lemma 2.1). According to Lemma 2.2 we infer that at least one of the symmetry groups of the polynomials  $f$  and  $g$  is nontrivial, say, the symmetry group  $G(f)$ . Then applying Lemma 2.4 we deduce that  $(G, \cdot)$  satisfies either

$$(xy)z = (zy)x \quad \text{or} \quad (xy)z = (yx)z \quad \text{or} \quad (xy)z = (xz)y.$$

If  $(G, \cdot)$  satisfies  $(xy)z = (zy)x$ , then using Proposition 2.5 we infer that  $(G, \cdot)$

is a nontrivial affine space over  $\text{GF}(5)$  but such algebras are polynomially equivalent to a commutative groupoid which contradicts the assumption.

Since  $P_2(G, \cdot) = 3$ , applying Proposition 2.6 the identity  $(xy)z = (yx)z$  does not hold in the groupoid  $(G, \cdot)$ .

Analogously, using Proposition 2.11 we conclude that the identity  $(xy)z = (xz)y$  also does not hold in  $(G, \cdot)$  which completes the proof of the Theorem.

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